Bernstein-Markov type inequalities and discretization of norms

András Kroó *
Alfréd Rényi Institute of Mathematics, Budapest, HUNGARY

In the past 15-20 years the problem of discretization of uniform and L^q norms in various finite dimensional spaces attracted a considerable attention. In case of $L^q, 1 \leq q < \infty$ norms this problem is usually referred to as the *Marcinkiewicz-Zygmund type problem.* On the other hand in function spaces with uniform norm the terms *norming sets* or *optimal meshes* are usually used in the literature. Historically the first discretization result was given by S.N. Bernstein in 1932 for the uniform norm: for any trigonometric polynomial t_n of degree $\leq n$ and any $0 = x_0 < x_1 < ... < x_N < 2\pi = x_{N+1}$ with $\max_{0\leq j\leq N}(x_{j+1} - x_j) \leq \frac{2\sqrt{\tau}}{n}$, $0 < \tau < 2$ we have

$$\max_{x\in[0,2\pi]} |t_n(x)| \leq (1+\tau) \max_{0\leq j\leq N} |t_n(x_j)|. \tag{1}$$

Thus the uniform norm of trigonometric polynomials of degree $\leq n$ can be discretized with accuracy τ using $N \sim \frac{n}{\sqrt{\tau}}$ properly chosen nodes. A standard substitution $x = \cos t$ leads to an extension of (1) for algebraic polynomials when $\max_{0\leq j\leq N}(\arccos x_{j+1} - \arccos x_j) \leq \frac{2\sqrt{\tau}}{n}$. The first result on the discretization of the $L^q, 1 < q < \infty$ norm is due to Marcinkiewicz and Zygmund who verified in 1937 that for any univariate trigonometric polynomial t_n of degree at most n and every $1 < q < \infty$ we have

$$\int |t_n|^q \sim \frac{1}{n} \sum_{s=0}^{2n} \left| t_n\left(\frac{2\pi s}{2n+1}\right)\right|^q. \tag{2}$$

Above relations provide an effective tool for the discretization of the L^q norms of univariate trigonometric and algebraic polynomials which is widely applied in the study of the convergence of Fourier series, Lagrange and Hermite interpolation, positive quadrature formulas, scattered data interpolation, etc. Numerous generalizations of the Marcinkiewicz-Zygmund type inequalities were subsequently given for weighted L^q norms, multivariate polynomial on sphere, ball and general convex domains, exponential polynomials.

In terms of the methods used for the discretization the following main general approaches appear in the literature:

1) *Functional analytic methods*
2) *Probabilistic methods*
3) *Methods based on Bernstein-Markov type inequalities*

While above approaches complement each other in different ways and make it possible to cover various cases it should be mentioned that in contrast to the functional analytic and probabilistic methods the **Bernstein-Markov approach** always yields *explicit* discretization nodes. The main goal of the present talk is to present a survey of some recent discretization results based on various classic and new Bernstein-Markov type inequalities. We will give an overview of corresponding Bernstein-Markov type inequalities including some recently established estimates for exponential sums, as well. Subsequently it will be shown how these Bernstein-Markov type inequalities yield new discretization results.

*Supported by the Hungarian National Research, Development and Innovation Fund NKFIH - OTKA Grant K128922