On the value of the fifth maximal projection constant

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Abstract

Let X be a Banach space over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Let $Y \subset X$ be a subspace. By $\mathcal{P}(X,Y)$ denote the set of all linear and continuous projections from X onto Y, recalling that an operator $P\colon X \to Y$ is called a *projection* onto Y if $P|_Y = \mathrm{Id}_Y$. We define the *relative projection constant* of a subspace Y of a space X by

$$\lambda(Y,X) := \inf\{\|P\| : \ P \in \mathcal{P}(X,Y)\}.$$

Now we can define the *absolute projection constant* of Y by

$$\lambda(Y) := \sup\{\lambda(Y,X) : Y \subset X\}. \tag{0.1}$$

The ultimate goal of researchers in this area is to determine the exact value of *maximal absolute projection constant*, which is defined by

$$\lambda_{\mathbb{K}}(m) := \sup\{\lambda(Y) : \ \dim(Y) = m\}.$$

In 1960, B.Grünbaum conjectured that $\lambda_{\mathbb{R}}(2) = \frac{4}{3}$ (see [6]), and only in 2010, B. Chalmers and G. Lewicki proved it (see [2]) and that was the only known nontrivial case. Recently, we have provided exact values of $\lambda_{\mathbb{K}}(m)$ in cases where the maximal equiangular tight frame exists in $\mathbb{K}^m$. There are numerous examples of complex maximal ETFs, for example, for $m \in \{1, \ldots, 17, 19, 24, 28, 35, 48\}$ (see, e.g., [5]). In fact, it is conjectured that there is a complex maximal ETF in every dimension (Zaurner's conjecture [7]). Unlike in the complex case, real maximal ETFs seem to be rare objects. The only known cases are for m equal to 2, 3, 7 and 23. A lot of the community believes that these are all real cases where maximal ETFs exist. In other cases, the determination of the constant $\lambda_{\mathbb{R}}(m)$ seems to be difficult. Numerical experiments conducted by B. L. Chalmers (and unfortunately unpublished) suggest that $\lambda(5) \approx 2.06919$. In this talk, relying on a new construction of certain mutually unbiased equiangular tight frames, we will provide the lower bound for $\lambda_{\mathbb{R}}(5)$ and present some arguments that it may be its true value.

References

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